Multi solitary waves to stochastic nonlinear Schrödinger equations

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Contents

I. Motivations

We consider the focusing mass-(sub)critical stochastic nonlinear Schrödinger equation:

$$dX = i\Delta X dt + i|X|^{p-1}X dt - \mu X dt + X dW(t), \qquad \text{(SNLS)}$$
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- Focusing mass-(sub)critical nonlinearity: $1 < p \le 1 + \frac{4}{d}$.
- W is the colored Wiener process

$$W(t,x) = \sum_{k=1}^{N} i\phi_k(x)B_k(t), \quad x \in \mathbb{R}^d, \ t \ge 0.$$

$\phi_k \in C_b^\infty(\mathbb{R}^d)$, $\mathrm{Re} W = 0$ (*conservative case*),
B_k are independent real-valued Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$.

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  XdW is taken in the sense of controlled rough path.

- $\mu(x) = \frac{1}{2} \sum_{k=1}^{N} \phi_k^2(x)$, $x \in \mathbb{R}^d$ (Conservation law of mass verified).

Brief review:

1. Subcritical case
   - ◇ GWP: [de Bouard-Debussche CMP'99, SAA'03], [Brzeźniak-Millet Potential Anal.'14], [Barbu-Röckner-Z., J.Nonlinear.Sci.'14, Nonlinear Anal.'16]...
   - ◇ Strichartz estimates: [Brzeźniak-Liu-Zhu JFA'21], [Z. SIMA'22]
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2. Defocusing critical case
   - ◇ GWP and scattering are more difficult, the standard energy method fails.
   - ◇ It was a main conjecture for deterministic NLS: [Bourgain JAMS'99], [Tao et al. AOM'08]
   - ◇ GWP, scattering: [Fan-Xu Anal.PDE'21], [Z. AAP'23+]
   - ◇ Optimal control problem [Z. PTRF'20]

3. Focusing critical case: blow-ups and solitons, more delicate

- ◇ [Bang et al Phys.Rev.'94, Appl.Anal.'95], [Rasmussen et al Phys.Letters'95]

  Physical study in $d = 1, 2$. 2-d collapse is important in the nonlinear optics.

- ◇ [Debussche-Menza Phys.D'02, Appl. Math. Lett.'02], [Millet et al SPDE'21, IMA J.Appl.Math.'21]

  Investigation of stochastic blow-ups by numerical experiments.

- ◇ [de Bouard-Debussche AOP'02, PTRF'05]

  Supercritical case: *conservative* noise accelerates blow-up with positive probability.

- ◇ [Barbu-Röckner-Z. JDE'16]

  (Super)critical case: *non-conservative* noise prevents blow-up with high probability.

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Question: quantitative description of blow-up/soliton profiles ?

Deterministic model: Nonlinear Schrödinger equation (NLS):

$$i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u = 0, \quad u(0) = u_0 \in H^1(\mathbb{R}^d).$$

- Invariance: under the translation, scaling, phase rotation, Galilean transform. L^2-norm is invariant under $u(t,x) \to \lambda^{-\frac{d}{2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ $\Rightarrow$ mass-critical.

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- Pseudo-conformal invariance:

$$u(t,x) \to (-t)^{-\frac{d}{2}} u(\frac{1}{-t}, \frac{x}{-t}) e^{-i\frac{|x|^2}{4t}}, \qquad t \neq 0.$$

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- Conservation laws:

$$\text{Mass}: \quad M(u)(t) := \int_{\mathbb{R}^d} |u(t)|^2 dx = M(u_0),$$

$$\text{Energy}: \quad E(u)(t) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t)|^2 dx - \frac{d}{2d+4} \int_{\mathbb{R}^d} |u(t)|^{2+\frac{4}{d}} dx = E(u_0),$$

$$\text{Momentum}: \quad Mom(u) := \mathrm{Im} \int_{\mathbb{R}^d} \nabla u \bar{u} dx = Mom(u_0).$$

Ground state Q:

- The unique positive radial solution to the nonlinear elliptic equation

$$\Delta Q - Q + Q^{1+\frac{4}{d}} = 0.$$

- Threshold for GWP and blow-up [Weinstein CMP'82]:
  - ◇ Solutions exist globally in the subcritical mass case $\|u_0\|_{L^2} < \|Q\|_{L^2}$;
  - ◇ Singularity can be formed in the (super)critical mass case $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$.

1. Critical mass regime $\|u_0\|_{L^2}^2 = \|Q\|_{L^2}^2$:

- Solitary wave

$$W(t,x) := w^{-\frac{d}{2}} Q\left(\frac{x-ct}{w}\right) e^{i(\frac{1}{2}c\cdot x - \frac{1}{4}|c|^2 t + w^{-2}t + \vartheta)},$$

where $c \in \mathbb{R}^d$ is the velocity, $w > 0$, $\vartheta \in \mathbb{R}$.

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- Pseudo-conformal blow-up solutions

$$S_T(t,x) = (w(T-t))^{-\frac{d}{2}} Q\left(\frac{x-x^*}{w(T-t)}\right) e^{-\frac{i}{4}\frac{|x-x^*|^2}{T-t} + \frac{i}{w^2(T-t)} + i\vartheta},$$

where $x^* \in \mathbb{R}^d$ is the blow-up point, $T, w > 0$, $\vartheta \in \mathbb{R}$.

  ◇ S_T blows up at the point x^* at time T, with the speed $\|S_T\|_{H^1} \sim (T-t)^{-1}$.

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- Pseudo-conformal transform:

$$S_T(t,x) = \mathcal{C}_T(W)(t,x) := \frac{1}{(T-t)^{\frac{d}{2}}} W\left(\frac{1}{T-t}, \frac{x}{T-t}\right) e^{-i\frac{|x|^2}{4(T-t)}},$$

where $t \neq T$, $x^* = c$.

2. Small supercritical mass regime $\|Q\|_{L^2}^2 < \|u_0\|_{L^2}^2 \leq \|Q\|_{L^2}^2 + \varepsilon$:

- Bourgain-Wang type blow-up solutions (Bourgain-Wang'97):

$$u(t) - S_T(t) - z(t) \to 0, \quad as\ t\ \to T.$$

  - ◇ Behave as a sum of a pseudo-conformal blow-up S_T and a regular profile z.

- Log-log blow-up solutions $\|\nabla u\|_{L^2} \sim \sqrt{\frac{\log|\log(T-t)|}{T-t}}$.

  - ◇ See [Merle-Raphaël GAFA'03, Invent.Math.'04, AOM'05, JAMS'06]

3. Large mass case $\|u_0\|_{L^2}^2 > \|Q\|_{L^2}^2$:

- **Mass Quantization Conjecture** [Bourgain GAFA'00], [Merle-Raphaël CMP'05]
  Mass of blow-up solutions is quantized at each singularity:

$$|u(t)|^2 \rightharpoonup \sum_{1\leq i\leq K} m_i \delta_{x_i} + |z^*|^2, \quad \text{with } m_i \in [\|Q\|_{L^2}^2, \infty)$$

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- **Soliton Resolution Conjecture** [Soffer ICM 2006], [Cazenave Lecture notes'20]
  Global solutions behave like a sum of solitons plus a scattering remainder:

$$v \to \sum_{1\leq i\leq K} W_i + \widetilde{z}^*$$

  "Note that the multi-soliton results fit in the soliton resolution conjecture. However, there is no dispersive term. It seems that there is no available results of a solution in the form of a multi-soliton plus a dispersive term. (Except in the integrable case.)"

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  Question 2: Non-pure multi-solitons ?

- **Uniqueness Open Problem** [Martel ICM 2018]:

  whether uniqueness holds for multi-solitons s.t.

$$\|v(t) - \sum_{k=1}^{K} W_k(t)\|_{H^1} = o(1), \quad \text{as } t \to \infty \text{ ?}$$

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  Previous results:

  - ◇ Uniqueness of critical mass (single-bubble) blow-up solutions [Merle Duke.Math.J.'93].
  - ◇ Qualitative behavior of critical mass solutions [Dodson Ann.PDE'23, arXiv:2104.11690].
  - ◇ Uniqueness of multi-solitons with the decay rate $(1/t)^N$ for N large enough [Côte-Friederich Comm.PDE'21].

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  Question 3: Uniqueness in the low asymptotical regime ?

Recent works:

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- Multi-bubble Bourgain-Wang blow-up solutions and non-pure multi-solitons [Röckner-Su-Z. Tran. AMS'23+]
  - Multi-bubble blow-ups:

$$\|e^{-W}X - \sum_{k=1}^{K} S_k - z\|_{L^2} + (T-t)\|e^{-W}X - \sum_{k=1}^{K} S_k - z\|_{\Sigma} = \mathcal{O}((T-t)^{4+});$$

  - Non-pure multi-solitons:

$$\|u(t) - \sum_{k=1}^{K} W_k(t) - \widetilde{z}(t)\|_{\Sigma} = \mathcal{O}(t^{-5-})$$

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  - ◇ Non-pure multi-solitons:

$$\|u(t) - \sum_{k=1}^{K} W_k(t) - \widetilde{z}(t)\|_{\Sigma} = \mathcal{O}(t^{-5-})$$

- Refined uniqueness of multi blow-ups and solitons [Cao-Su-Z. ARMA'23]
  - ◇ Multi-bubble blow-up solutions:

$$\|v(t) - \sum_{k=1}^{K} S_k(t)\|_{H^1} = \mathcal{O}((T-t)^{0+}), \quad for\ t \to T;$$

  - ◇ Multi-solitons:

$$\|u(t) - \sum_{k=1}^{K} W_k(t)\|_{H^1} = \mathcal{O}(t^{-2-}), \quad for\ t \to \infty.$$

- Question 4: Stochastic multi-solitons ?

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Difficulty:

◇ Absence of pseudo-conformal invariance:

Unlike in the deterministic case [Merle CMP'90], stochastic solitons cannot be obtained from the stochastic blow-up solutions in [Su-Z. JFA'23], [Su-Z. arXiv: 2012.14037];

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◇ Instability of multi-solitons:

Unlike the stable scattering in [Herr-Röckner-Z. CMP'19], [Z. AAP'23+], multi-solitons are unstable under perturbations of initial data.

II. Stochastic multi-solitons

- **L^2-(sub)critical SNLS**

$$dX = i\Delta X dt + i|X|^{p-1} X dt - \mu X dt + \sum_{k=1}^{N} X G_k dB_k(t),$$

where

- ◇ $1 < p \leq 1 + \frac{4}{d}$, $d \geq 1$, $\{B_k\}$ and μ are as in previous sections.
- ◇ $G_k(t,x) = i\phi_k(x) g_k(t)$, $\{\phi_k\} \subseteq C_b^\infty(\mathbb{R}^d, \mathbb{R})$, $\{g_k\}$ are $\{\mathscr{F}_t\}$-adapted processes with paths in $C^\alpha(\mathbb{R}^+, \mathbb{R})$ that are controlled by $\{B_k\}$, $\alpha \in (1/3, 1/2)$.
- ◇ $X(t)G_k(t)dB_k(t)$ is taken in the sense of controlled rough paths.

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- ◇ $X(t)G_k(t)dB_k(t)$ is taken in the sense of controlled rough paths.

- **Solitary waves**

$$R_k(t,x) := Q_{w_k^0}(x - v_k t - x_k^0) e^{i(\frac{1}{2} v_k \cdot x - \frac{1}{4}|v_k|^2 t + (w_k^0)^{-2} t + \theta_k^0)},$$

where

- ◇ $Q_w(x) := w^{-\frac{2}{p-1}} Q\left(\frac{x}{w}\right)$;
- ◇ $w_k^0 > 0$, $\theta_k^0 \in \mathbb{R}$, $x_k^0 \in \mathbb{R}^d$, $v_k \in \mathbb{R}^d \setminus \{0\}$, $v_j \neq v_k$ if $j \neq k$, $1 \leq k \leq K$.

Assumption

- (H1)' $\lim_{|x|\to\infty} |x|^2|\partial_x^\nu \phi_l(x)| = 0, \quad \nu \neq 0, \ 1 \leq l \leq N.$
- (H2)' $\{g_l\}$ is $\{\mathscr{F}_t\}$-adapted continuous processes and controlled by B.M. $\{B_l\}$. Moreover, ϕ_l and g_l satisfy Case (I)' or Case (II)':
  - ◇ Case (I)': $g_l \in L^2(\mathbb{R}^+)$, $\mathbb{P} - a.s.$ and

$$\sum_{|\nu|\leq 4} |\partial^\nu \phi_l(x)| \lesssim e^{-|x|}.$$

  - ◇ Case (II)': $\mathbb{P}$-a.s., $g_l \in L^2(\mathbb{R}^+)$, $\exists \nu_* \in \mathbb{N}$ s.t.

$$\sum_{|\nu|\leq 4} |\partial^\nu \phi_l(x)| \lesssim |x|^{-\nu_*},$$

and

$$\int_t^\infty g_l^2 ds \log\left(\int_t^\infty g_l^2 ds\right)^{-1} \lesssim t^{-2}, \quad \text{for } t \text{ large.}$$

Theorem [Röckner-Su-Z. PTRF'23]

For $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist $T_0 = T_0(\omega)$ large and $X_*(\omega) \in H^1$, such that there exists a solution $X(\omega) \in C([T_0, \infty); H^1)$ to SNLS, $X(\omega, T_0) = X_*(\omega)$ and

$$\|e^{-W_*(t)}X(t) - \sum_{k=1}^{K} R_k(t)\|_{H^1} \lesssim \int_t^\infty s\phi^{\frac{1}{2}}(\delta s)ds, \quad t \geq T_0.$$

Here

- $\diamond$ $W_*(t,x) = -\sum_{l=1}^{N} \int_t^\infty i\phi_l(x)g_l(s)dB_l(s)$.
- $\diamond$ $\phi(x) := e^{-|x|}$ or $|x|^{-\nu_*}$ in Case (I)' or Case (II)', respectively.

Remark

Temporal asymptotical rate can be of either exponential and polynomial type, respectively, in Case (I)' and Case (II)', related to spatial decay rate of noise.

III. Outline of proof

Proof is based on two types of rescaling transforms and modulation method:

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Step 1

- Recaling transformation:

$$u = e^{-W_*(t)} X(t),$$

gives a new equation

$$iu_t + (\Delta + b_* \cdot \nabla + c_*)u + |u|^{p-1}u = 0$$

where $u(T_0) = e^{-W_*(T_0)} X_0,\ b_* = 2\nabla W_*,\ c_* = \sum_{j=1}^{d} (\partial_j W_*)^2 + \Delta W_*.$

Step 2

- Geometrical decomposition

$$u(t,x) = \sum_{k=1}^{K} \widetilde{R}_k(t,x) + \varepsilon(t,x) \ (=: \widetilde{R}(t,x) + \varepsilon(t,x))$$

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- Soliton profiles:

$$\widetilde{R}_k(t,x) := Q_{w_k(t)}\left(x - v_k t - \alpha_k(t)\right) e^{i(\frac{1}{2} v_k \cdot x - \frac{1}{4}|v_k|^2 t + (w_k^0)^{-2} t + \theta_k(t))}.$$

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$$\widetilde{R}_k(t,x) := Q_{w_k(t)}\,(x - v_k t - \alpha_k(t))\, e^{i(\frac{1}{2} v_k \cdot x - \frac{1}{4}|v_k|^2 t + (w_k^0)^{-2} t + \theta_k(t))}.$$

- Modulation parameters: $\mathcal{P}_k := (\alpha_k, \theta_k, w_k)$, $\mathcal{P}_k(T) = (x_k^0, \theta_k^0, w_k^0)$.

Step 2

- Geometrical decomposition

$$u(t,x)=\sum_{k=1}^{K}\widetilde{R}_k(t,x)+\varepsilon(t,x)\ (=:\widetilde{R}(t,x)+\varepsilon(t,x))$$

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$$\widetilde{R}_k(t,x):=Q_{w_k(t)}\left(x-v_kt-\alpha_k(t)\right)e^{i(\frac{1}{2}v_k\cdot x-\frac{1}{4}|v_k|^2t+(w_k^0)^{-2}t+\theta_k(t))}.$$

- Modulation parameters: $\mathcal{P}_k:=(\alpha_k,\theta_k,w_k)$, $\mathcal{P}_k(T)=(x_k^0,\theta_k^0,w_k^0)$.
- Remainder ε: $\varepsilon(T)=0$.

Step 2

- Geometrical decomposition

$$u(t,x) = \sum_{k=1}^{K} \widetilde{R}_k(t,x) + \varepsilon(t,x) \ (=: \widetilde{R}(t,x) + \varepsilon(t,x))$$

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- Remainder ε: $\varepsilon(T) = 0$.
- Orthogonality:

$$\mathrm{Re}(\nabla \widetilde{R}_k, \varepsilon) = 0, \quad \mathrm{Im}(\widetilde{R}_k, \varepsilon) = 0, \quad \mathrm{Re}((\Lambda_k \widetilde{R}_k - \frac{i}{2} v_k \cdot y_k \widetilde{R}_k(t)), \varepsilon) = 0,$$

where $\Lambda_k := \frac{2}{p-1} I_d + y_k \cdot \nabla$, $y_k(t) = x - v_k t - \alpha_k(t)$.

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- Control of modulation equations:

$$\begin{aligned} Mod_k(t) :=& |\dot{w}_k(t)| + |\dot{\alpha}_k(t)| + |\dot{\theta}_k(t) - (w_k^{-2}(t) - (w_k^0)^{-2})| \\ \lesssim& \|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + e^{-\delta_2 t}. \end{aligned}$$

Step 3

- Local mass

$$I_k(t) := \int |u(t,x)|^2 \varphi_k(t,x) dx,$$

where $\{\varphi_k\}$ are suitable localized functions.

- Local momentum:

$$M_k(t) := \mathrm{Im} \int \nabla u(t,x) \bar{u}(t,x) \varphi_k(t,x) dx.$$

- Lyapunov type functional

$$G(t) := 2E(u(t)) + \sum_{k=1}^{K} \{((w_k^0)^{-2} + \frac{|v_k|^2}{4}) I_k(t) - v_k \cdot M_k(t)\}.$$

- Coercivity type control

$$\|\varepsilon(t)\|_{H^1}^2 \lesssim (\int_t^\infty \frac{1}{s} \|\varepsilon(s)\|_{H^1}^2 ds + (\int_t^\infty \frac{1}{s} \|\varepsilon(s)\|_{H^1}^2 ds)^2) + (\int_t^\infty B_*(s)(\|\varepsilon\|_{H^1}^2 + \phi(\delta_1 s)) ds + e^{-\delta_2 t}).$$

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$$+ (\int_t^\infty B_*(s)(\|\varepsilon\|_{H^1}^2 + \phi(\delta_1 s))ds + e^{-\delta_2 t}).$$

Step 4 Bootstrap arguments.

Thank you very much